

Note on a q -modified central limit theorem

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Abstract

A q -modified version of the central limit theorem due to Umarov *et al.* affirms that q -Gaussians are attractors under addition and rescaling of certain classes of strongly correlated random variables. The proof of this theorem rests on a nonlinear q -modified Fourier transform. By exhibiting an invariance property we show that this Fourier transform does not have an inverse. As a consequence, the theorem falls short of achieving its stated goal.

Keywords: q -modified nonlinear Fourier transform, q -modified central limit theorem, nonextensive statistical mechanics

1 Introduction

Since over two decades work by Tsallis and co-workers [1, 2, 3] has drawn attention to problems in statistical physics where Boltzmann-Gibbs statistics is not or not directly applicable. These problems typically involve long-range interactions and correlations. Tsallis initiated an approach to such systems that is often referred to as *q-modified* or *nonextensive* statistical mechanics. One of its characteristics is the occurrence of so-called *q*-Gaussian probability distributions; these are defined by

$$G_q(x) = \frac{C_q}{[1 + (q - 1)x^2]^{\frac{1}{q-1}}}, \quad (1.1)$$

where q is an arbitrary real number, C_q is the normalization constant, and the domain of definition is the real x -axis (for $q \geq 1$) or the part thereof where $1 + (q - 1)x^2 \geq 0$ (for $q < 1$). For $q \rightarrow 1$ equation (1.1) reduces to an ordinary Gaussian.

In *q*-modified statistical mechanics it is expected that *q*-Gaussians arise naturally when one considers sums of *strongly* correlated variables, at least for a certain class of correlations; this by analogy to ordinary Gaussians, that describe sums of sufficiently *weakly* correlated variables. Therefore, the observation of *q*-Gaussians, whether in nature or in numerical simulations, would lend support to the applicability of *q*-modified statistical mechanics.

There have recently been attempts to provide a theoretical basis for this special status attributed to *q*-Gaussians. One attempt has consisted in numerically generating and adding up random variables with strong correlations of some well-controlled kind, and determining the distribution of the sum. In two examples studied by Thistleton *et al.* [4, 5] and by Moyano *et al.* [6] the sum distribution is numerically virtually indistinguishable from a *q*-Gaussian and each was initially believed by their authors to be one. In subsequent analytical work Hilhorst and Schehr [7] showed, however, that in fact the analytic expressions of these sum distributions are unrelated to *q*-Gaussians. It was pointed out [7, 8], moreover, that correlations between variables may be engineered in such a way that the sum of N of them, scaled and in the limit $N \rightarrow \infty$, has any desired distribution.

The question therefore is not whether a sum of strongly correlated variables *can* have a *q*-Gaussian distribution¹, but whether under the operation of addition and scaling of random variables such *q*-Gaussians appear as *attractors*, at least for certain classes of correlations wide enough to be of interest for physics.

¹This is indeed possible; among the examples are those of references [9, 8].

A second attempt to mathematically consolidate the special role attributed to the q -Gaussian probability law is due to Umarov *et al.* [10]. These authors, in the wake of earlier conjectures [11, 6], prove a q -modified central limit theorem (q -CLT) in which the limit functions are q -Gaussians². We recently expressed concern [8] about two points in the proof of this theorem:

(i) First, let $x_1, x_2, \dots, x_n, \dots$ denote the sequence of random variables whose scaled partial sums we wish to study. Then in order for the q -CLT to apply, the first N of these variables, for all $N = 1, 2, \dots$, must be correlated according to a certain condition stated explicitly in [10] and termed “ q -independence”. This property indeed holds, according to [10], for a conveniently chosen sequence of random variables that are themselves, individually, q -Gaussian distributed. However, to our knowledge, no other example that would illustrate the theorem has been exhibited so far. It is therefore not sure that it is at all possible to fulfill the conditions of the theorem in any nontrivial way.

(ii) Secondly, the proof of the theorem is based on the use of a q -modification of the ordinary Fourier transform. This q -FT is nonlinear and, again, tailored to q -Gaussian distributions: when applied to a q -Gaussian, it produces (up to scaling factors) a q' -Gaussian where $q'(q)$ is a known function. It was observed in reference [8] that this q -FT does not have an inverse. This observation is the subject of the present note.

We will first show here the proof of noninvertibility hinted at in reference [8] and which is based on a counterexample. Whereas a single counterexample may be mathematically sufficient to make a point, we next ask whether one can perhaps obtain an invertible transformation by restricting the domain of action of the q -FT to a suitable subspace of probability distributions. The answer turns out to be negative. By explicit construction we exhibit families of functions all having the same q -Fourier transform; and we show that the q -Gaussians themselves are part of such families.

Sections 2 and 3 deal with the definition and the non-invertibility of the q -Fourier transform, respectively. In section 4 we present a brief discussion and conclusion.

2 A q -modified Fourier transform

Let $f(x) \geq 0$ be an integrable function on the real axis. Umarov *et al.* [10] define its nonlinear q -Fourier transform $\hat{f}_q(\xi)$ as

$$\hat{f}_q(\xi) = \int_{-\infty}^{\infty} dx \frac{f(x)}{[1 - (q-1) i \xi x f^{q-1}(x)]^{\frac{1}{q-1}}}, \quad (2.1)$$

²Extensions of the theorem to the multivariate case [12] and to q -modified α -stable Lévy distributions [13] have appeared since.

where ξ is real and from now on $q > 1$; furthermore here and henceforth $f^{q-1}(x) \equiv [f(x)]^{q-1}$. For $q \rightarrow 1^+$ expression (2.1) reduces to the standard Fourier transform. If $f(x)$ is a probability distribution, as will be the case, the normalization condition

$$\int_{-\infty}^{\infty} dx f(x) = 1 \quad (2.2)$$

has to be imposed.

The q -FT (2.1) has since its introduction been discussed on various occasions by the original authors and others [14, 15, 16, 17, 18]. In what follows we will show that it is not invertible, even when restricted to the space of probability distributions.

3 Noninvertibility of the q -FT

3.1 An example

We begin by examining a particular case ([8], footnote [30]).

Example 1. Let us consider

$$f(x) = \begin{cases} \left(\frac{\lambda}{x}\right)^{\frac{1}{q-1}} & x \in (a, b), \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

where $0 < a < b$ and $\lambda > 0$. Due to the normalization (2.2) the constant λ can be expressed in terms of the interval end points a and b , which yields

$$\lambda = \left[\frac{q-1}{q-2} \left(b^{\frac{q-2}{q-1}} - a^{\frac{q-2}{q-1}} \right) \right]^{-(q-1)}. \quad (3.2)$$

Substituting (3.1) in (2.1) and doing the integral using the normalization of f leads to

$$\hat{f}_q(\xi) = \frac{1}{[1 - (q-1) i \xi \lambda]^{\frac{1}{q-1}}}. \quad (3.3)$$

This is the same result for any interval (a, b) that satisfies (3.2) with a fixed λ . Hence equations (3.1)-(3.2) define a one-parameter family of normalized functions $f(x)$ all having the same q -Fourier transform (3.3). This counterexample shows that the q -FT is not invertible on the space of probability distributions.

3.2 An invariance property

For physicists it is important to know that the above example is not an isolated case that might somehow be eliminated by suitably restricting the space of functions. We will therefore show how other examples may be constructed, many of them corresponding to physically reasonable probability distributions. The construction rests on the fact that for a large class of functions $f(x)$ the expression

$$\lambda(x) = x f^{q-1}(x) \quad (3.4)$$

is not invertible to a single-valued function $x(\lambda)$. Let us rewrite the q -FT of equation (2.1) as

$$\hat{f}_q(\xi) = \int_{-\infty}^{\infty} d\lambda \frac{F(\lambda)}{[1 - (q-1) i \xi \lambda]^{\frac{1}{q-1}}} \quad (3.5)$$

in which

$$\begin{aligned} F(\lambda) &= \int_{-\infty}^{\infty} dx \delta(x f^{q-1}(x) - \lambda) f(x) \\ &= \sum_i \left| \frac{d}{dx} [x f^{q-1}(x)] \right|_{x=x_i}^{-1} f(x_i), \end{aligned} \quad (3.6)$$

where the sum runs through the set $\{x_i\}$ of the solutions of $x f^{q-1}(x) = \lambda$. Obviously, if two distinct functions $f_1(x)$ and $f_2(x)$, when substituted in (3.6), lead to the same $F(\lambda)$, then by (3.5) they will have the same q -FT. *This invariance leads to the nonuniqueness of the inverse q -FT.*

3.3 A class of symmetric functions

An investigation of this invariance in its most general form would probably begin by classifying the $f(x)$ according to the number of terms that they engender in the sum on i in equation (3.6). It is not needed for our purpose to embark on so broad an enterprise. We will study here a more limited but important class of functions $f(x)$ which contains, in particular, the q -Gaussians. None of the restrictions on $f(x)$ to be adopted below is essential; each can be overcome by a little more work.

Let us consider for convenience a symmetric function, $f(x) = f(-x)$, that is finite in the origin, $f(0) < \infty$. Its values for $x > 0$ (for $x < 0$) then determine $F(\lambda)$ for $\lambda > 0$ (for $\lambda < 0$) and we have $F(\lambda) = F(-\lambda)$. We may therefore limit our analysis to $x, \lambda > 0$. Let us furthermore restrict our attention to those $f(x)$ for which $\lambda(x) = x f^{q-1}(x)$ is monotonic on $(0, \infty)$

except for passing through a single maximum at $x = x_m$; and let $\lambda_m \equiv \lambda(x_m)$. We note that the monotonicity condition is not a very severe one and is satisfied, in particular, by the q -Gaussians (1.1).

Let now $f(x)$ be in the class delimited above. Full generality not being our purpose, let us suppose that $1 < q \leq 2$. In that case the integrability of $f(x)$ implies that we must have

$$\lambda(0) = \lambda(\infty) = 0. \quad (3.7)$$

Let us denote by $x_-(\lambda)$ and $x_+(\lambda)$ the inverses of $\lambda(x)$ on the intervals $[0, x_m]$ and $[x_m, \infty)$, respectively. The pair $x_{\pm}(\lambda)$ is an alternative representation of $f(x)$. Using that $dx_+/d\lambda < 0 < dx_-/d\lambda$ and that $f(x) = (\lambda/x)^{\frac{1}{q-1}}$ we obtain from (3.6) for $F(\lambda)$ the expression

$$F(\lambda) = \frac{dx_-}{d\lambda} \left(\frac{\lambda}{x_-(\lambda)} \right)^{\frac{1}{q-1}} - \frac{dx_+}{d\lambda} \left(\frac{\lambda}{x_+(\lambda)} \right)^{\frac{1}{q-1}}, \quad (3.8)$$

which we cast in the final form

$$F(\lambda) = \frac{q-2}{q-1} \lambda^{\frac{1}{q-1}} \frac{d}{d\lambda} \left[x_-^{\frac{q-1}{q-2}}(\lambda) - x_+^{\frac{q-1}{q-2}}(\lambda) \right] \quad (3.9)$$

where $0 \leq \lambda \leq \lambda_m$. Starting from (2.2) and replacing x by λ as the variable of integration, we find along similar lines that the normalization of a symmetric function $f(x)$ can be expressed as

$$2 \int_0^{\lambda_m} d\lambda F(\lambda) = 1. \quad (3.10)$$

Equation (3.9) shows that the invariance can now be expressed as follows. The function $F(\lambda)$ does not change if in (3.9) we substitute $x_{\pm}(\lambda) \mapsto \tilde{x}_{\pm}(\lambda)$ with

$$\begin{aligned} \tilde{x}_-^{\frac{q-1}{q-2}}(\lambda) &= x_-^{\frac{q-1}{q-2}}(\lambda) + G(\lambda) + G_-, \\ \tilde{x}_+^{\frac{q-1}{q-2}}(\lambda) &= x_+^{\frac{q-1}{q-2}}(\lambda) + G(\lambda) + G_+, \end{aligned} \quad (3.11)$$

where G_+ and G_- are constants and $G(\lambda)$ is an arbitrary function. Equation (3.5) shows that this substitution does not change the q -Fourier transform $\hat{f}_q(\xi)$ and (3.10) shows that it does not change the normalization of $f(x)$. The function $G(\lambda)$ should satisfy certain rather mild constraints coming from the fact that we want the pair $\tilde{x}_{\pm}(\lambda)$ to be again the representation of a probability distribution $\tilde{f}(x)$. This distribution $\tilde{f}(x)$, if it exists, can therefore be constructed in the following two steps:

1. Invert $\tilde{x}_{\pm}(\lambda)$ to a function $\tilde{\lambda}(\tilde{x})$.

2. Solve $\tilde{f}(x)$ from $x\tilde{f}^{q-1}(x) = \tilde{\lambda}(x)$, which gives

$$\tilde{f}(x) = \left(\frac{\tilde{\lambda}(x)}{x} \right)^{\frac{1}{q-1}}. \quad (3.12)$$

The result is a function $\tilde{f}(x)$ different from $f(x)$ but which has the same q -FT.

3.4 Three more examples

We will exhibit below three examples that result from an implementation of the procedure of section 3.3. We have not sought to exploit the full freedom offered by the occurrence of the arbitrary function $G(\lambda)$ in (3.11), but replaced it with a single parameter.

Example 2. Let us take for $f(x)$ the q -Gaussian $G_q(x)$ of equation (1.1). For this function direct calculation gives

$$x_{\pm}(\lambda) = \frac{C_q^{q-1} \pm [C_q^{2(q-1)} - 4(q-1)\lambda^2]^{\frac{1}{2}}}{2\lambda(q-1)}, \quad 0 < \lambda \leq \lambda_m, \quad (3.13)$$

with

$$\lambda_m = \frac{1}{2}(q-1)^{-\frac{1}{2}}C_q^{q-1}. \quad (3.14)$$

Expression (3.13) has the properties

$$x_{\pm}(0) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{and} \quad x_{\pm}(\lambda_m) = x_m = (q-1)^{-\frac{1}{2}}. \quad (3.15)$$

Let us choose $G(\lambda) = A$ where $A \geq 0$ is a parameter, and $G_+ = G_- = 0$ in (3.11). Then

$$\tilde{x}_{\pm}(\lambda) = \left(x_{\pm}^{\frac{q-2}{q-1}}(\lambda) + A \right)^{\frac{q-1}{q-2}}. \quad (3.16)$$

In expression (3.16) the $x_{\pm}(\lambda)$ are given by (3.13) and therefore the $\tilde{x}_{\pm}(\lambda)$ can be inverted to a function $\tilde{\lambda}(\tilde{x})$. The result is

$$\tilde{\lambda}(\tilde{x}) = \frac{2\lambda_m y}{1+y^2}, \quad y \equiv (q-1)^{\frac{1}{2}} \left(\tilde{x}^{\frac{q-2}{q-1}} - A \right)^{\frac{q-1}{q-2}}. \quad (3.17)$$

We will now indicate the A dependence explicitly and write $\tilde{f}_A(x)$ for the function $\tilde{f}(x)$ represented by (3.17). Using (3.12) we find from (3.17)

$$\begin{aligned} \tilde{f}_A(x) &= \left(\frac{\tilde{\lambda}(x)}{x} \right)^{\frac{1}{q-1}} \\ &= \frac{C_q \left(x^{\frac{q-2}{q-1}} - A \right)^{\frac{1}{q-2}}}{x^{\frac{1}{q-1}} \left[1 + (q-1) \left(x^{\frac{q-2}{q-1}} - A \right)^{2\frac{q-1}{q-2}} \right]^{\frac{1}{q-1}}}, \end{aligned} \quad (3.18)$$

valid in the domain $A \leq x^{\frac{q-2}{q-1}} < \infty$, whereas $\tilde{f}_A(x) = 0$ for $0 \leq x^{\frac{q-2}{q-1}} \leq A$ and $\tilde{f}_A(-x) = \tilde{f}_A(x)$. We have not indicated the q -dependence of this family explicitly. The $A = 0$ member of family (3.18) is the original q -Gaussian $G_q(x)$. By construction all $\tilde{f}_A(x)$ have the same q -FT, independently of A . Hence we have constructed a one-parameter family containing a q -Gaussian and for which the q -FT has no inverse.

Example 3. We consider the special case $q = \frac{3}{2}$ of equation (3.18). The reason is that, whereas different statements concerning the q -FT occurring in the literature may have different domains of validity on the q axis, virtually all of them apply for $1 \leq q \leq 2$. Hence a value in the middle of this interval is among the most relevant ones. For $q = \frac{3}{2}$ equation (3.18) simplifies to

$$\tilde{f}_A(x) = \frac{C_{\frac{3}{2}} (x^{-1} - A)^{-2}}{x^2 \left[1 + \frac{1}{2} (x^{-1} - A)^{-2} \right]^2}. \quad (3.19)$$

We will return to this example in our conclusion.

Example 4. We consider the special case $q = 2$ of equation (3.18). The limit $q \rightarrow 2$ is singular. To take this limit we replace A by a parameter a defined as

$$A = -\frac{q-2}{q-1} \log a. \quad (3.20)$$

The conditions $A \geq 0$ and $1 < q \leq 2$ that we imposed above now require that $a \geq 1$. Substituting (3.20) for A in (3.18) and taking the limit $q \rightarrow 2^-$ we get, with an obvious change of notation,

$$\tilde{f}_a(x) = \frac{C_2 a}{1 + a^2 x^2}, \quad (3.21)$$

which is equal to $aG_2(ax)$. Hence we have found here that the 2-FT of $aG_2(ax)$ is independent of a , for $a \geq 1$. This independence is in fact valid for all $a > 0$ and is of course easily demonstrated by direct calculation.

4 Conclusion

The authors of reference [10] and later work on the q -FT have certainly been aware of the necessity for that transformation to have an inverse. Reference [14] deals exclusively with this issue. However, the authors limit their investigation to the question of whether *a q -Gaussian obtained under the q -FT has a unique preimage in the subspace of q -Gaussians*, that is, to asking if $q'(q)$ has an inverse. The answer to the question thus narrowed down is affirmative but has little bearing on the problem of the invertibility of the q -FT on a full space of functions.

Our example 3 appears³ in the overview by Tsallis [18], who pays ample attention there to the problem of invertibility. Tsallis proposes to select a specific value of the parameter A , hence a specific member of the family $f_A(x)$, by prescribing the second moment of that function. Our view, quite apart from other questions that such a procedure may raise, is that this is an *ad hoc* fix covering the one-parameter case. It does not lift the degeneracy in cases – whose existence we have made plausible – where a single q -Fourier transform is associated with a many-parameter family or with a family that depends on an arbitrary function, that is, on a continuum of parameters. The presentation of our derivation in section 3 had, precisely, the purpose of showing the extent of the invertibility problem.

In summary, we have shown that, when considered on a reasonably large space of functions, the q -modified Fourier transformation employed in the work by Umarov *et al.* [10] does not possess an inverse. As a consequence, that work remains unsuccessful in its attempt of showing that q -Gaussians are attractors under addition and rescaling.

The numerical search for q -Gaussians has dealt with statistical models that are less amenable to analytic treatment than those of references [4, 5, 6, 7] mentioned above. This search has in particular focused on the logistic map and the Hamiltonian Mean Field Model (see [8] for references), where various types of scaled partial sums have been proposed as candidates for q -Gaussian distributed variables. In the future still other models will no doubt be examined with the same purpose. We leave numerical work of that nature out of the present discussion because, first, strictly mathematically speaking we have nothing to say about it; and, secondly, the interpretation of the numerical results has in each case been controversial. The only relevant corollary of the present note is that fitting numerical data with q -Gaussians cannot be justified on the basis of a q -central limit theorem.

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